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THE SET OF IRREDUCIBLE GRAPHS FOR THE PROJECTIVE PLANE IS FINITE*

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In 1930 Kuratowski proved that a graph does not embed in the real plane $\mathbb{R}^2$ if and only if it contains a subgraph homeomorphic to one of two graphs, K_5 or $K_{3,3}$. For positive integer n, let $I_n(P)$ denote a smallest set of graphs whose maximal valency is n and such that any graph which does not embed in the real projective plane contains a subgraph homeomorphic to a graph in $I_n(P)$ for some n. Glover and Huneke and Milgram proved that there are only 6 graphs in $I_3(P)$, and Glover and Huneke proved that $I_n(P)$ is finite for all n. This note proves that $I_n(P)$ is empty for all but a finite number of n. Hence there is a finite set of graphs for the projective plane analogous to Kuratowski's two graphs for the plane.

1. Introduction

For a 2-dimensional manifold M let $I(M)$ denote a smallest set of graphs such that every graph which does not embed in M contains a subgraph homeomorphic to one of the graphs in $I(M)$ (see Section 2). Let $\{I_n(M) \mid n \in N\}$ be a partition of $I(M)$ such that $K \in I_n(M)$ if and only if $K \in I(M)$, and the maximal valency of a vertex in K is n. There has been limited success in describing $I(M)$ or even $I_n(M)$. The known results are, for $\mathbb{R}^2$ the real plane and P the real projective plane:

Theorem 1.1. (Kuratowski [4].) *$I(\mathbb{R}^2)$ contains exactly two graphs, K_5 (the complete graph on 5 vertices) and $K_{3,3}$ (the complete 3,3 bipartite graph.)*

Theorem 1.2. (Glover and Huneke [1]; Milgram [5].) *$I_3(P)$ contains exactly 6 graphs.*

Theorem 1.3. (Glover and Huneke [2].) *$I_n(P)$ is finite for every natural number n.*

Theorem 1.4. (Glover, Huneke and Wang [3].) *$I(P)$ contains at least 103 graphs.*

Other brief lists of graphs in $I(M)$ for other surfaces have been published. Also Vollmerhaus [6] announced some general results in 1969 which would imply the

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main result of this paper; the proofs of these "results" are unavailable. The main result of this paper is:

Theorem 3.4. *$I(P)$ is finite.*

Standard results and the pertinent properties of P are listed as propositions in Section 2 along with definitions of terms used in this paper. Section 3 states and proves the main result from lemmas which are presented with proofs in Section 4 (for Theorem 3.1) and Section 5 (for Theorem 3.2).

2. Fundamentals

This section contains all the technical terminology and notation used in this paper. It also contains several propositions each of which is either immediate from the definitions, or is a result of topology or topological graph theory. References are offered for the propositions. All except one are stated without proof.

A graph is a finite 1-dimensional CW complex, i.e., a finite set of points called vertices and a finite set of 1-cells called edges which contain vertices as endpoints only, each end point a vertex; a graph has the topology of a CW complex. An edge in a graph K with endpoints v_1, v_2 will be denoted $[v_1, v_2]$. The valency of a vertex v in K is the number of times v is an endpoint for edges of K. A subgraph of a graph K is a graph whose vertices and edges are in K. A graph K satisfies a property minimally if K but no proper subgraph of K satisfies the property. A graph is called an arc if it is homeomorphic to an edge $[v_1, v_2]$ with $v_1 \neq v_2$. If L_1 and L_2 are subgraphs of a graph K then A is an arc from L_1 to L_2 provided A is an arc in K such that $A \cap L_1$ and $A \cap L_2$ are the two endpoints of A; also L_1 and L_2 will be called disjoint if $L_1 \cap L_2 = \emptyset$, or equivalently if L_1 and L_2 have no common vertices. Finally a graph K will be called connected provided K is connected as a topological space, and, for natural number n, will be called n-connected provided removing any $n-1$ vertices (and adjacent edges) leaves a connected graph (equivalently by Menger's Theorem, there are n arcs between any two vertices which meet pairwise only at their endpoints.)

Notation. Let $A \subset X$ be topological spaces.

(i) $X-A$ denotes the complement of $A \subset X$.

(ii) cls A denotes the closure of A in X.

(iii) A is said to separate in X two points of X provided the points are in distinct components of $X-A$.

Let K be a graph, $L \subset K$ a subgraph.

(iv) $K-L$ denotes the subgraph of K composed of all edges and all vertices of K which are not in L. Note that all vertices belonging to edges of $K-L$ also belong to $K-L$ and that $K-L = \text{cls}\,(K-L)$ as topological spaces.

(v) K/L denotes the quotient space, which is a graph. In particular, the vertices of K/L are the vertices of K which are not in L together with one additional vertex denoted $\{L\}$.

(vi) The star of L, written $\mathrm{st}(L)$, denotes the subgraph of K composed of L together with all edges with at least one endpoint in L. Again note that all vertices belonging to edges of $\mathrm{st}(L)$ also belong to $\mathrm{st}(L)$.

(vii) For natural number n, let K_n denote the complete graph on n vertices (the 1-skeleton of the $(n-1)$-simplex).

(viii) For natural numbers n, m, let $K_{n,m}$ denote the complete n, m bepartite graph (the joint of discrete sets of size n and m respectively.)

(ix) K is called a θ graph provided K is homeomorphic to $K_{2,3}$.

(x) L is a k-graph of K provided there exists a subgraph L' of K, L a subgraph of L', such that either (a) L is homeomorphic to $K_{2,3}$, L' is homeomorphic to $K_{3,3}$, and one of the vertices in L' containing 3 edges of L' is not in L, or (b) L is homeomorphic to K_4, L' is homeomorphic to K_5 and one of the vertices in 4 edges in L' is not in L. or (c) L is homeomorphic to K_4, L' contains an arc A with L'/A homeomorphic to K_5, and A is disjoint from L (see [2]).

Two other technical types of graphs, a B-graph and a 2-connected (L_1, L_2) partition, are defined when used in Section 4 and Section 5 respectively.

Let $\mathbf{R}^2$ denote the Euclidean plane, S^2 denote the 2-dimensional sphere and P the real projective plane.

If X and Y are topological spaces, a (continuous) map $f: X \to Y$ is called an embedding if the corestriction $f: X \to f(X)$ is a homeomorphism. We say that X embeds in Y provided there exists an embedding $f: X \to Y$.

A 2-dimensional topological manifold is called a surface. If M is a surface then: (i) $A \subset M$ is contractible in M provided there is a 2-disc D with $A \subset D \subset M$; (ii) a cycle $C \subset M$ is essential provided C is not contractible in M; (iii) $I(M)$ denotes the set of graphs which have no valency 2 vertex and minimally do not embed in M (observe, each homeomorphy class of all graphs which minimally do not embed in M contains a unique representative in $I(M)$;) (iv) for natural number n, $I_n(M)$ denotes the subset of $I(M)$ such that $K \in I_n(M)$ provided K contains a vertex of valency n, and each vertex of K has valency $\leq n$.

In the following propositions assume M is a surface.

Proposition 2.1. (Lemma 2.2 in [3].) *P contains no pair of disjoint essential cycles.*

Proposition 2.2. (See Lemma 2.5 in [3].) *If L is a k-graph of a graph K, and if $\varphi: K \to M$ is an embedding, then there is a simple cycle C in L with $\varphi(C)$ essential in M.*

Proposition 2.3. *Let $\varphi: K_{2,3} \to P$ be an embedding. Then either $\varphi(K_{2,3})$ (and every cycle in $\varphi(K_{2,3})$) is contractible or there exists a unique cycle $L \subset K_{2,3}$ with $\varphi(L)$ contractible.*

Proof. If $\varphi(K_{2,3})$ is not contractible then there is a cycle $L_1 \subset K_{2,3}$ such that $\varphi(L_1)$ is essential in P. Hence $P-\varphi(L_1)$ is a disc so each component of $P-\varphi(K_{2,3})$ is a disc. The Euler Characteristic of P is 1 and also is $5-6+n$, where n is the number of components of $P-\varphi(K_{2,3})$, since $K_{2,3}$ has 5 vertices and 6 edges. Hence $n=2$. If there is an edge e in $K_{2,3}$ such that $P-\varphi(K_{2,3}-e)$ still contains 2 components then for the unique cycle $L \subset (K_{2,3}-e)$, $P-\varphi(L)$ contains 2 components so is not essential. If for every edge e in $K_{2,3}$ $P-\varphi(K_{2,3}-e)$ has only one component, then every edge of $K_{2,3}$ is in the boundary of both components of $P-\varphi(K_{2,3})$ which is impossible since $K_{2,3}$ has a vertex with odd valency. Hence there is a cycle L in $K_{2,3}$ with $\varphi(L)$ contractible. If L_1 is a cycle, $L \neq L_1 \subset K_{2,3}$ with $\varphi(L_1)$ contractible, then $\varphi(L \cup L_1)$ is contractible and $L_1 \cup L = K_{2,3}$. Hence the result.

Proposition 2.4. (See Lemma 4.5 in [2].) *Let K be a graph without valency 2 vertices, let e be an edge of K, let L_1 be a 2-connected subgraph of K not containing e, let L_2 be a connected component of $K-\mathrm{st}\,L_1$ which does not intersect e. If K does not embed in M but $\varphi: K-e \to M$ is an embedding such that $\varphi(L_1)$ is contractible in M, then there is a k-graph of K disjoint from L_2.*

Proposition 2.5. (See Proposition 2.7 in [2].) *Let $K \in I(M)$. Then every cycle in K contains at least 3 vertices and each vertex in K has valency at least 3.*

Proposition 2.6. (See Theorem 3.1. in [2].) *Suppose that $K \in I(P)$ contains a θ graph which is disjoint from a k-graph of K. Then K contains no more than 16 vertices.*

Proposition 2.7. (See Lemma 4.4 in [2].) *Suppose that $K \in I(P)$ does not contain a θ graph which is disjoint from a k-graph of K. Then K is 3-connected.*

3. Main results

The purpose of this paper is to establish an upper bound on the valency of a vertex in a graph in $I(P)$, for then by Glover and Huneke [2] (see Theorem 1.3) it follows that $I(P)$ is finite. As in [2], we have no particular concern for the particular upper bound on the valency of a vertex in a graph in $I(P)$.

Theorem 3.1. *Let $K \in I_n(P)$ be a graph containing no disjoint θ subgraphs. Then $n \leqslant 26$.*

Proof. Assume $n \geqslant 27$ and seek a contradiction. Let $v \in K$ be a vertex with valency n. By Proposition 2.7, K is 3-connected. By Lemma 4.4 there is a cycle $L \subset K-\mathrm{st}\,v$ such that $L \cap \mathrm{st}\,v$ contains at least 13 vertices. Lemmas 4.5 and 4.6

imply K contains disjoint θ subgraphs, contradicting the hypothesis of the theorem. Hence the result follows.

Theorem 3.2. *Let $K \in I_n(P)$ be a graph which contains disjoint θ graphs but does not contain a θ graph disjoint from a k-graph of K. Then $n \leqslant 14$.*

Proof. Let v be a vertex of K with the valency of $v = n$. If $K - \text{st } v$ is nonplanar then $n \leqslant 12$ by Lemma 5.1. If $K - \text{st } v$ is planar then by Lemmas 5.5 and 5.4, st v contains no more than $6+8$ edges so $n \leqslant 14$. Hence the result follows.

Theorem 3.3. *Let $K \in I_n(P)$ be a graph which contains a θ graph disjoint from a k-graph of K. Then $n \leqslant 15$.*

Proof. By Proposition 2.6, K contains no more than 16 vertices so $n \leqslant 15$. Hence the result follows.

Theorem 3.4. *$I(P)$ is finite.*

Proof. For each natural number n, $I_n(P)$ is finite by Theorem 1.3 and for $n \geqslant 27$, $I_n(P)$ is empty by Theorems 3.1, 3.2 and 3.3. However $\{I_n(P) \mid n = 1, 2, \ldots\}$ is a partition of $I(P)$. Hence the result follows.

It should be remarked that the bound $n \leqslant 15$ in Theorem 3.3 can be improved to $n \leqslant 8$, by inspecting the proof of Proposition 2.6, and there exists an example in this case of a graph with a vertex of valency 8. The most complete list of graphs in $I(P)$ known to the authors is Wang's list [3] of 103 graphs in $I(P)$ in which the upper bound on the valency is 8 (or is 6 for graphs in $I(P)$ satisfying the hypothesis of Theorems 3.1 or 3.2) and the upper bound on the number of vertices is 13 (or is 12 for graphs in $I(P)$ satisfying the hypothesis of Theorems 3.1 or 3.2). Also, all graphs in $I(P)$ which do not satisfy the hypotheses of Theorems 3.1 and 3.2 are characterized in [2] and listed in [3].

4. Graphs without disjoint θ's

The lemmas of this Section 4 establish Theorem 3.1, an upper bound on the valency of a vertex in a graph in $I(P)$ which does not contain two disjoint θ graphs.

Lemma 4.1. *Let K be a connected graph, let $\varphi : K \to P$ be an embedding, and let D be a component of $P - \varphi(K)$. Then $\text{cls } D - D$ is connected.*

Proof. For every embedding of a connected graph in the plane, each component of the complement of the graph has a connected boundary. Hence for every embedding of a connected graph in a contractible subset of any surface, the boundary of each component of the complement is connected. Hence if cls $D-D$ is not connected then each component of cls $D-D$ must be essential in P so P would have to contain disjoint essential cycles, contradicting Proposition 2.1. Hence the result follows.

Lemma 4.2. *Let K be a connected graph without cut points, let M be a surface, let $\varphi: K \to M$ be an embedding, let D be a component of $M-\varphi(K)$, let L be the subgraph of K defined by $L=\varphi^{-1}(\text{cls } D-D)$ and let v be a vertex of L. Then: (i) v is in at least 2 edges of L; (ii) if v is cut point of L then for every cycle $L_1 \subset L$ containing v, $\varphi(L_1)$ is essential; and (iii) if v is a cut point of L or is in at least 3 edges of L_1 then there is a cycle C_1 in $D \cup \{\varphi(v)\}$ containing $\varphi(v)$ and there is a cycle L_1 in L containing v such that C_1 and $\varphi(L_1)$ are essential.*

Proof. If v is in fewer than 2 edges of L then $D \cup \varphi \text{ st}(v \cap L)$ is a neighborhood of $\varphi(v)$ so a neighborhood of v in K intersects fewer than 2 edges of K; hence v has valency less than 2 in K so K is either not connected or has a cut point, a contradiction. Hence the result for (i) follows.

Suppose v is a cut point for L and L_1 is a cycle with $v \in L_1 \subset L$ such that $\varphi(L_1)$ is not essential. Then $\varphi(L_1)$ bounds a disc D_1 in M. If $D \cap D_1 \neq \emptyset$ then $D \subset D_1$ and $D \neq D_1$ so v is a cut point of $L_1 \cup \varphi^{-1}(D_1)$ and hence v is a cut point of K since D_1 is homeomorphic to $\boldsymbol{R}^2$. If $D \cap D_1 = \emptyset$ then $D \cup D_1 \cup \varphi(L_1)$ is a neighborhood of $\varphi(L_1)-\varphi(v)$ and there is a disc $D_2 \supset D_1$ such that D_2 is a neighborhood of $\varphi(L_1)$ and $\varphi(v)$ is a cut point of $\varphi(K) \cap D_2$, hence v is a cut point of K, a contradiction. Hence the result for (ii) follows.

If every cycle in $D \cup \{\varphi(v)\}$ is contractible in M, then there is an open disc D_1 in M containing $D \cup \varphi(v)$ and $\varphi(v)$ separates in $\varphi(L)$ a connected subgraph $L_2 \subset L_1$ with $\varphi(L_2) \subset D_1$ from another point in $\varphi(L)$; hence v is a cut point of K, a contradiction. Hence there is an essential cycle C in $D \cup \varphi(v)$ with $\varphi(v) \in C$. Since C can be chosen arbitrarily close to the boundary of D, and the pointwise limit of essential cycles is essential, the desired cycle L_1 exists in L. Hence the result follows.

Definition. A graph K is called a B graph provided K is a necklace of polygons, that is, provided K contains a cycle L and set of arcs α from L to L such that $K = L \cup \bigcup_{A \in \alpha} A$ and for each $A \in \alpha$ there is an arc A^* in L such that $A \cup A^*$ is a cycle, and for A_1, A_2 distinct in α, the corresponding arcs A_1^*, A_2^* in L have $A_1^* \cap A_2^* \subset A_1 \cap A_1^*$ (the end points of A_1), and $A_1 \cap A_2 \subset L$.

Lemma 4.3. *Let $e=[v, v_1]$ be an edge in a 2-connected graph K. Let $\varphi: K-e \to P$ be an embedding. Let D be the component of $P-\varphi(K-\text{st } v)$ which contains $\varphi(v)$. Let L be the subgraph of $K-\text{st } v$ such that $\varphi(L) \cup D = \text{cls } D$. Then every vertex in st v is in $L \cup e$, and L is homeomorphic to $K_5 - K_{3,2}$ (the wedge of two circles), or L is a B graph.*

Proof. Since $\varphi(\text{st } v - e) \subset \varphi(K-e) \cap \text{cls } D = \varphi L \cup \varphi \text{ st } v - e)$, every vertex in st v is in $L \cup e$. By Lemma 4.1, cls $D - D = \varphi(L)$ is connected so L is connected. To characterize L, consider three cases.

Case 1. Assume L has two cut points, v_1 and v_2, and seek a contradiction. There are three subgraphs L_i, $i = 1, 2, 3$ of L, each connected containing an edge, such that $L = \bigcup_{i=1}^{3} L_i$ and $L_i \cap L_{i+1} = v_i$. Since each vertex in L is in at least 2 edges of L, v_2 and L_3 can be chosen such that there is a cycle L_4 in L_3 with $v_2 \in L_4$. Since K is 3-connected, $K - \text{st } v$ is 2-connected and by Lemma 4.2 there is a cycle C in $D \cup \varphi(v_1)$ such that C, as well as $\varphi(L_4)$, is essential in P. However C and $\varphi(L_4)$ are disjoint which contradicts Proposition 2.1. Hence the result in Case 1 follows.

Case 2. Assume L is not 2-connected, and show that L is homeomorphic to $K_5 - K_{2,2}$. Since L contains a unique cut point v_1, L contains two cycles L_1, L_2 such that $L_1 \cap L_2 = v_1$. It remains in Case 2 to show that $L = L_1 \cup L_2$. If $L \neq L_1 \cup L_2$ then there is an arc A in L from $L_1 \cup L_2$ to $L_1 \cup L_2$. For some $i = 1$ or 2, $A \cup L_i$ contains a vertex $v_2 \neq v_1$ with v_2 in 3 edges of L, and by Lemma 4.2, $D \cup \varphi(v_2)$ contains an essential cycle C. Also for $j = 1$ and 2, $\varphi(L_j)$ is essential in P so C is an essential cycle in P disjoint from an essential cycle $\varphi(L_j)$, which contradicts Proposition 2.1. Hence the result in Case 2 follows.

Case 3. Assume L is 2-connected and show that L is a B graph. Assume L is not a B graph. Then L is not a cycle so contains a vertex in 3 edges of L and by Lemma 4.2 there is a cycle $L_1 \subset L$ such that $\varphi(L_1)$ is essential in P. Since L is 2-connected, $L = L_1 \cup \bigcup_{A \in \mathfrak{a}} A$ where $\mathfrak{a}$ is a set of arcs in L from L_1 to L_1. Every vertex in 3 edges of L is in L_1 since otherwise P would contain an essential cycle disjoint from the essential cycle $\varphi(L_1)$ by Lemma 4.2. Hence distinct arcs in $\mathfrak{a}$ intersect only in L_1. By Proposition 2.3, for each arc $A \in \mathfrak{a}$ there is a unique cycle L_A in $L_1 \cup A$ such that $\varphi(L_A)$ is contractible in P; $A \subset L_A$ since $\varphi(L_1)$ is not contractible in P. If there are two distinct arcs A_1, A_2 in $\mathfrak{a}$ with $A_1 \cap L_1 = A_2 \cap L_1$ and if $\mathfrak{a}$ contains a third arc A_3, $A_1 \neq A_3 \neq A_2$, then two of the 3 cycles L_{A_i} $i = 1, 2, 3$ contain a common arc A. If there are arcs A_i, $i = 4, 5, 6$ such that the two vertices in $A_4 \cap L_1$ separate in the cycle L_1 a vertex of $A_5 \cap L_1$ from a vertex of $A_6 \cap L_1$, then L_{A_4} and a cycle L_{A_i} for $i = 5$ or 6 contain a common arc Since L is assumed not to be a B graph, this shows that there are two distinct arcs A_7, A_8 in $\mathfrak{a}$ such that $L_{A_7} \cap L_{A_8}$ contains an arc so $L_{A_7} \cup L_{A_8}$ is a θ graph. Since $\varphi(L_{A_i})$ is contractable in P for $i = 7$ and 8, $\varphi(L_{A_7} \cup L_{A_8})$ does not contain an essential cycle by Proposition 2.3. Hence $L_{A_7} \cup L_{A_8}$ contains a cycle L_2 such that $P - \varphi(L_2)$ has two components, D_1, D_2 such that D_1 is a disc, and $D_1 \cap \varphi(L_{A_7} \cup L_{A_8}) \neq \emptyset$. Since $\varphi(L_1)$ is not contractible, $\varphi(L_1) \not\subset D_1 \cup \varphi(L_{A_7} \cup L_{A_8}) = \text{cls } D_1$ so $\varphi(L_1) \cap D_2 \neq \emptyset$. Now D_1 and D_2 are disjoint open subsets of P, $D_i \cap \text{cls } D \neq \emptyset$ so $D_i \cap D \neq \emptyset$ for $i = 1, 2$; hence $D \subset D_1 \cup D_2$ so D is not connected, a contradiction. Hence the result follows.

Lemma 4.4. *Let $e=[v,v_1]$ be an edge in a 3-connected graph K, such that $K-e$ embeds in P and assume the valency of v is n. Then there is a simple cycle in $K-\mathrm{st}\,v$ which contains at least $(n-1)/2$ vertices of* st v.

Proof. Let L be the subgraph of $K-\mathrm{st}\,v$ given by the hypothesis of Lemma 4.3 from some embedding of $K-e$ in P. Observe that L contains at least $n-1$ vertices of st v. If L is homeomorphic to $K_5-K_{2,2}$ (the wedge of two circles) then one of the cycles in L contains $(n-1)/2$ vertices of st v; hence the result. Thus assume L is a B graph. Let L_1 be a cycle in L with a maximum number of vertices of st v such that $L=l_1\cup\bigcup_{A\in\alpha}A$ where α is a set of arcs from L_1 to L_1 which pairwise intersect in L_1. If L_1 contains at most two vertices such that each is in at least 3 edges of L, then α contains at most 2 arcs so the result is immediate. If L_1 contains at least three vertices each in at least 3 edges of L, then for each arc $A\in\alpha$ there is a unique arc A^* in L_1 with the same end points as A which contains endpoints of arcs in α only at $A\cap A^*$; observe that the number of vertices in $A^*\cap\mathrm{st}\,v$ is not greater than the number of vertices in $A\cap\mathrm{st}\,v$ by the maximality condition on L_1. Hence at least half of the vertices of $L\cap\mathrm{st}\,v$ are in L_1. Hence the result follows.

Lemma 4.5. *If v is a vertex of K and $K-\mathrm{st}\,v$ is a B graph then K embeds in P.*

Proof. It is sufficient to show there is an embedding $\varphi\colon K-\mathrm{st}\,v\to P$ such that $\varphi(K-\mathrm{st}\,v)\subset\mathrm{cls}\,D$ for one of the components D of $P-\varphi(K-\mathrm{st}\,v)$. For then φ would extend to an embedding $\bar\varphi:K\to P$ with $\bar\varphi(\mathrm{st}\,v)\subset\mathrm{cls}\,D$. Because $K-\mathrm{st}\,v$ is a B-graph, there is a cycle $L\subset K-\mathrm{st}\,v$ and a set of arcs α from L to L in $K-\mathrm{st}\,v$ which pairwise meet in L such that $K-\mathrm{st}\,v=L\cup\bigcup_{A\in\alpha}A$, and for each $A\in\alpha$ there is an arc A^* in L with the same endpoints as A such that A^* only intersects arcs in α at its endpoints, and such that $A_1^*\neq A_2^*$ if $A_1\neq A_2$. Now let $\varphi:K-\mathrm{st}\,v\to P$ be an embedding with $\varphi(L)$ an essential cycle and $\varphi(A\cup A^*)$ a contractible cycle for each $A\in\alpha$. The components of $P-\varphi(K-\mathrm{st}\,v)$ are precisely the discs bounded by $\gamma(A\cup A^*)$ for each $A\in\alpha$ and one additional disc whose boundary is $\varphi(K-\mathrm{st}\,v)$. Hence the result follows.

Lemma 4.6. *Let K be a 3-connected graph and v a vertex on K such that $K-\mathrm{st}\,v$ is not a B graph. Let L be a cycle in $K-\mathrm{st}\,v$ such that $L\cap\mathrm{st}\,v$ contains at least* 13 *distinct vertices. Then K contains disjoint θ subgraphs.*

Proof. If there is a connected graph $L_1\subset K-\mathrm{st}\,v$ such that $L_1\cap L=\emptyset$ and $L\cap\mathrm{st}\,(L_1)$ contains a set V of 3 distinct vertices, then one, L_2, of the three arcs in L from V to V contains more than 3 vertices in st v other than the end points of L_2 so st $v\cup L_2$ contains a θ graph which is disjoint from a θ graph in $L_1\cup L-L_2$; hence the result follows.

Thus assume $A_0 \cap L = A_1 \cap L$ for every two distinct arcs A_0, A_1 in $K - \mathrm{st}\, v$ from L to L such that $A_0 \cap A_1 \not\subset L$. If A_0, A_1 are distinct arcs in $K - \mathrm{st}\, v$ from L to L such that $A_0 \cap L = A_1 \cap L$, then L_3, one of the 2 arcs in L from $A_0 \cap L$ to $A_0 \cap L$, contains at least 6 vertices of st v other than its end points so a θ graph in $L_3 \cap \mathrm{st}\, v$ is disjoint from a θ graph in $A_0 \cup A_1 \cup L - L_3$; hence the result. Thus assume $K - \mathrm{st}\, v = L \cup \bigcup_{A \in \alpha}$ such that α is a set of arcs from L to L which meet pairwise on L at most at one vertex. $K - \mathrm{st}\, v$ is not a B graph by assumption, so there are arcs A_2, A_3, A_4 in α such that the two vertices of $A_2 \cap L$ separate in the cycle L a vertex of $A_3 \cap L$ and a vertex of $A_4 \cap L$. There are four arcs L_i, $i = 4, 5, 6, 7$ in L which pairwise intersect at most at one vertex such that $L = \bigcup_{i=4}^{7} L_i$ and for each j, $4 \geq j \leq 7$, there is a θ graph in $L \cup \bigcup_{i=2}^{4} A_i - L_j$. Also for some j, $4 \leq j \leq 7$, L_j contains at least 3 vertices of st v other than the end points of L_j, so $L_j \cup \mathrm{st}\, v$ contains a θ graph. Hence K contains two disjoint θ graphs.

5. Graphs with disjoint θ's.

Graphs $K \in I_n(P)$ which contain a θ graph disjoint from a k-graph of K are characterized in [2] so a bound on n is given in Theorem 3.3. Hence it remains to study graphs with disjoint θ subgraphs but without a θ graph disjoint from a k-graph of K. The lemmas of this section 5 establish Theorem 3.2, an upper bound on the valency of a vertex in a graph in $I(P)$ with disjoint θ graphs but no θ graph disjoint from a k-graph.

Lemma 5.1. *Let $K \in I(P)$ be a graph which does not contain a θ subgraph disjoint from a k-graph of K. Let v be a vertex of K such that $K - \mathrm{st}\, v$ is nonplanar. Then valency of v is at most* 12.

Proof. Let $L_1 \in I(\boldsymbol{R}^2)$ and let $\psi : L_1 \to K$ be an embedding with $v \notin \psi(L_1)$. Let L_2 be the maximal connected subgraph of K which contains v and is disjoint from $\psi(L_1)$. Since K does not contain a θ graph disjoint from a k-graph of $\psi(L_1)$, $(\psi(L_1) \cup \mathrm{st}\, L_2)/L_2$ does not contain a θ graph disjoint from a k-graph of $\psi(L_1) \subset (\psi(L_1) \cup \mathrm{st}\, L_2)/L_2$. Hence for each vertex v_1 of L_1, at most two edges in st $\{L_2\}$ (equivalently, in st $L_2 - L_2$) are disjoint from $\psi(L_1 - \mathrm{st}\, v_1)$. However L_1 has at most 6 vertices so 12 is an upper bound for the valency of $\{L_2\}$ in K/L_2, which is the number of edges in st $L_2 - L_2$. If L_2 contains a cycle C, then the valency of $\{C\}$ in K/C is at least the valency of any vertex of C since each cycle in L_2 contains at least 3 edges with each vertex of valency at least 3 by Proposition 2.5. Each cycle in L_2/C retains at least 3 edges with each vertex of valency (in K/C) at least 3 since L_2 contains no θ graph. Hence the cycles in L_2 can be collapsed inductively changing L_2 into a subtree, T, of a resulting quotient graph of K such that the maximum valency of the vertices of T is at least the valency of v. The edges of T can then be collapsed inductively without reducing valency. Hence the valency of $\{L_2\}$ in K/L_2 is at least the valency of v and the result follows.

Definition. If L_1, L_2 are disjoint θ subgraphs of a graph K, then a 2-connected (L_1, L_2) partition of K is a pair of disjoint 2-connected subgraphs of K, L_3, L_4 such that $L_i \subset L_{i+2}$ (for $i = 1, 2$), every vertex of K is in $L_3 \cup L_4$, and every edge of K is in $L_3 \cup L_4$ or contains a vertex of L_3 and a vertex of L_4.

Lemma 5.2. *If K is a 3-connected graph with disjoint θ graphs L_1, L_2 then there is a 2-connected (L_1, L_2) partition of K.*

Proof. Let L_3 be a maximal 2-connected subgraph of $K - \operatorname{st} L_2$ which contains L_1; L_3 exists since if L_1, L_2 are disjoint θ graphs then L_1 is a 2-connected subgraph of $K - \operatorname{st} L_2$. Let L_4 be a maximal 2-connected subgraph of $K - \operatorname{st} L_3$ which contains L_2. For $i = 3$ or 4, if e is an edge in K with both end points in L_i then $e \in L_i$ by the maximality conditions defining L_i. It remains to show that $L_3 \cup L_4$ contains every vertex of K; assume v is a vertex of K not in $L_3 \cup L_4$ and seek a contradiction. Let $\mathcal{a}$ be the set of all arcs in K from v to $L_3 \cup L_4$. Since K is 3-connected, and $L_3 \cup L_4$ contains at least 3 vertices, $(L_3 \cup L_4) \cap \bigcup_{A \in \mathcal{a}} A$ contains at least 3 vertices, at least 2 of which must be in L_i for $i = 3$ or 4; let A_1, A_2 be two arcs in $\mathcal{a}$ such that $L_i \cap (A_1 \cup A_2)$ is a set of two vertices. Then L_i is a proper subgraph of a 2-connected graph in $L_i \cup A_1 \cup A_2 \subset K - \operatorname{st} L_{i-1}$ contradicting the maximality condition which defines L_i. Hence the result follows.

Lemma 5.3. *Let $K \in I(P)$ such that no θ subgraph is disjoint from a k-graph of K. Let L_1, L_2 be disjoint θ subgraphs of K. Then:* (i) *there exist L_3, L_4, a 2-connected (L_1, L_2) partition of K;* (ii) *for each $i = 3, 4$, K/L_i is planar;* (iii) *for each $i = 3$, 4, there is a simple cycle $C_i \subset L_i$ such that $L_i \cap E \subset C_i$, where $E = K - (L_3 \cup L_4)$, and there is an embedding $\varphi_i : L_i \to S^2$ with $\varphi_i(C_i)$ the boundary of a component of $S^2 - \varphi_i(L_i)$;* (iv) *for $\{i, j\} = \{3, 4\}$ and for v a vertex of C_i there exist two connected graphs A_{1v}, A_{2v} in C_j (each an arc or a vertex) such that $E \cap C_j \cap \operatorname{st} v \subset A_{1v} \cup A_{2v}$ and $(E - \operatorname{st} v) \cap C_j \cap (A_{1v} \cup A_{2v})$ is a subset of the set of end points of A_{1v} or A_{2v};* (v) *for every embedding $\varphi : C_3 \cup C_4 \cup E \to P$, either $\varphi(C_3)$ or $\varphi(C_4)$ is an essential cycle in P; and* (vi) *for every vertex v of K such that $K - \operatorname{st} v$ is planar, $v \in C_3 \cup C_4$.*

Proof. (i). By Proposition 2.7, K is 3-connected. Hence by Lemma 5.2, there exists a 2-connected (L_1, L_2) partition of K.

(ii). Let $1 \leq i \leq 4$. If K/L_i is nonplanar then there is a k-graph of K/L_i in $(K/L_i) - \operatorname{st}(L_i)$, which is also a k-graph of K disjoint from L_i since L_i is connected. However L_i contains a θ graph and K contains no θ graph disjoint from a k-graph of K by hypothesis. Hence the result (ii) follows.

(iii). Let $\{i, j\} = \{3, 4\}$, and let $\varphi_i : K/L_j \to S^2$ be an embedding. $L_i = K - \operatorname{st} L_j = (K/L_j) - \operatorname{st}\{L_j\}$ is 2-connected so each component of $S^2 - \varphi_i(L_i)$ is homeomorphic to $\mathbb{R}^2$ and has a boundary which is a simple cycle in $\varphi_i(L_i)$. Let C_i be the cycle in

L_i such that $\varphi_i(C_i)$ is the boundary of the component of $S^2-\varphi_i(L_i)$ which contains $\varphi_i(\{L_j\})$. Now $\varphi_i(L_i\cap\mathrm{st}\{L_j\})\subset\varphi_i(C_i)$ so $L_i\cap E=L_i\cap\mathrm{st}\{L_j\}\subset C_i$.

(iv). Assume A_{1v} and A_{2v} do not exist and seek a contradiction. Let e be an edge of L_i-C_i and let $\varphi_e:K-e\to P$ be an embedding. Observe that L_j is a connected component of $K-\mathrm{st}\,C_i$ which does not intersect e.

Now $\varphi_e(C_i)$ is essential in P since otherwise $\varphi_e(C_i)$ would be contractible so there would be a k-graph of K disjoint from L_j by Proposition 2.4. Hence $\varphi_e(C_j)$ is contractible by Proposition 2.1. To assume that A_{1v} and A_{2v} do not exist is to assume there are three edges, e_k, in st v with end points $v_k\in C_j$, $k=1, 3, 5$, and three edges e_k of $E-\mathrm{st}\,v$ with end points $v_k\in C_j$, $k=0, 2, 4$, such that the vertices $\{v_4\,|\,0\leq k\leq 5\}$ are 6 distinct vertices on which the cyclic order induced by C_j agrees with the natural order of the indices. Let L_5 be a θ graph in $C_j\cup e_1\cup e_3\cup e_5$ which contains $e_1\cup e_3\cup e_5$. By proposition 2.3, there is a cycle $C\subset L_5$ such that $\varphi_e(C)$ is contractible. $C\cup C_j$ is a θ graph in K which is a k-graph in $C_i\cup C_j\cup\bigcup_{i=0}^{5}e_i$, and $\varphi_e(C\cup C_j)$ is contractible. However φ_e embeds $C_i\cup C_j\cup\bigcup_{i=0}^{5}e_i$ into P so $\varphi_e(C\cup C_j)$ must not be contractible by Proposition 2.2. Hence the result (iv) follows.

(v). Let $M=S^2$ if $C_3\cup C_4\cup E$ is planar and $M=P$ otherwise. Assume $\varphi:C_3\cup C_4\cup E\to M$ is an embedding with $\varphi(C_3)$, $\varphi(C_4)$ contractible cycles in M and seek a contradiction by showing that K then embeds in M. Let $i=3$ or 4. Let D_1 be the component of $S^2-\varphi_i(C_i)$ which does not contain $\varphi_i(\{L_j\})$; cls D_i is a disc containing $\varphi_i(L_i)$ whose boundary is $\varphi_i(C_i)$. Let D_i' be the component of $M-\varphi(C_i)$ which is also a component of $M-\varphi(C_3\cup C_4\cup E)$; D_i' exists since $C_j\cup E$ is connected, and D_i is homeomorphic to $\mathbf{R}^2$ since otherwise $M=P$ and D_i' contains a simple cycle which is essential in P and disjoint from $\varphi(C_3\cup C_4\cup E)$ so $\varphi(C_3\cup C_4\cup E)$ is contractible in P, hence $C_3\cup C_4\cup E$ is planar and $M=S^2\neq P$, a contradiction. Let $\psi_i:\mathrm{cls}\,D_i\to\mathrm{cls}\,D_i'$ be a homeomorphism such that $\psi_i\circ\varphi_i\,|_{C_i}=\varphi\,|_{C_i}$. Define $\psi:K\to M$ by

$$\psi=\psi_3\varphi_3\,|_{L_3}\cup\psi_4\varphi_4\,|_{L_4}\cup\varphi\,|_{C_3\cup C_4\cup E}.$$

The function ψ is well defined 1 to 1 and continuous on three closed sets which cover K; hence $\psi:K\to M$ is an embedding, which contradicts $K\in I(P)$. Hence the result in (v) follows.

(vi). If $K-\mathrm{st}\,v$ is planar and $v\notin C_3\cup C_4$ then $C_3\cup C_4\cup E$ is planar which contradicts (v). Hence the result follows.

Lemma 5.4. *Let $K\in I(P)$ be a graph which contains disjoint θ graphs but which does not contain a θ graph disjoint from a k-graph of K. Let v be a vertex of K such that $K-\mathrm{st}\,v$ is planar. Then, for E defined by Lemma* 5.3, *$E\cap\mathrm{st}\,v$ contains at most* 8 *edges.*

Proof. The notation of the statement of Lemma 5.3 is assumed and used in this proof (e.g. $v\in C_3\cup C_4$.) Since the indices 3, 4 are symmetric in Lemma 5.3, assume without loss of generality that $v \in C_4$ and that $A_{1v} \cup A_{2v} \subset C_3$. Assume $E \cap \mathrm{st}\, v$ contains 9 edges and seek a contradiction. Then for $j = 1$ or 2, five edges of $E \cap \mathrm{st}\, v$ intersect A_{jv}. Without loss of generality, assume there are $m \geq 5$ distinct edges e_k, $1 \leq k \leq m$, in $E \cap \mathrm{st}\, v$ such that each contains a vertex v_k in A_{1v} and the linear order induced on $\{v_k \mid 1 \leq k \leq m\}$ by A_{1v} agrees with the natural order induced by the indices; observe that $\{v_k \mid 1 \leq k \leq 5\}$ is a set of 5 distinct vertices since K has no cycles containing only two edges. Let A be the arc in A_{1v} with end points $\{v_2, v_4\}$ and let L_5 be the θ graph $A \cup e_2 \cup e_3 \cup e_4$. Observe that K contains a θ graph, L_6, which is disjoint from L_5 since if not, then there would be at most two edges in $E - \mathrm{st}\, v$, and hence $C_3 \cup C_4 \cup (E - \mathrm{st}\, v)$ would embed in P with C_3 and C_4 contractible and $C_3 \cup C_4$ in the boundary of one component of the complement in P of the image of $C_3 \cup C_4 \cup (E - \mathrm{st}\, v)$, and hence $C_3 \cup C_4 \cup E$ would embed in P with C_3 and C_4 contractible, contradicting Lemma 5.3 (v). Let $\varphi_1 : K - e_3 \to P$ be an embedding. $\varphi_1(L_5 - e_3)$ is an essential cycle in P since otherwise L_6 would be a θ graph disjoint from a k-graph of K by Proposition 2.4, a contradiction of the hypothesis. It remains to seek a contradiction in each of 3 cases.

Case 1. Assume $\varphi_1(C_3)$ is essential. Then $\varphi_1(e_2 \cup e_4 \cup C_3 - A)$ is a contractible cycle in P by Proposition 2.1. Let D_1, D_2 be the two components of $P - \varphi_1(e_2 \cup e_4 \cup C_3 - A)$ with $\varphi_1(C_4) \cap D_1 \neq \emptyset$. Now, $C_3 \cup C_4 \cup E - \mathrm{st}\, L_5$ is connected since $C_4 \cap E$ contains a vertex other than v, so $\varphi_1((C_3 \cup C_4 \cup E) - (A \cup e_3)) \subset \mathrm{cls}\, D_1$. The end points of A, and v, are in $\mathrm{cls}\, D_2$ and $D_2 \cap \varphi_1(C_3 \cup C_4 \cup E - e_3) = \emptyset$ so there is an embedding $\varphi_2 : (C_3 \cup C_4 \cup E) \to P$ such that

$$\varphi_2 \big|_{(C_3 \cup C_4 \cup E) - (A \cup e_3)} = \varphi_1 \big|_{(C_3 \cup C_4 \cup E) - (A \cup e_3)}$$

and

$$\varphi_2(A \cup e_3) \subset \mathrm{cls}\, D_2$$

with $\varphi_2(A)$ in a contractible neighborhood of the arc $\varphi_1(C_3 - A)$. Now $\varphi_2(C_4) = \varphi_1(C_4)$ is disjoint from the essential cycle $\varphi_1(C_3)$ so is contractible, and $\varphi_2(C_3)$ is in a contractible subset of P so is contractible. Hence φ_2 contradicts Lemma 5.3 (v).

Case 2. Assume $\varphi_1(C_4)$ is essential; we show that K embeds in P, a contradiction. Let D be the component of $S^2 - \varphi_3(C_3)$ which intersects $\varphi_3(L_3 - C_3)$. The closed disc $\mathrm{cls}\, D$ contains $\varphi_3(L_3)$ and has boundary $\varphi_3(C_3)$. We know $\varphi_1(C_3)$ is contractible by Proposition 2.1, so let D_1, D_2 be the components of $P - \varphi_1(C_3)$ with D_1 contractible, and let $\psi : \mathrm{cls}\, D \to \mathrm{cls}\, D_1$ be a homeomorphism such that $\psi\varphi_3 \big|_{C_3} = \varphi_1 \big|_{C_3}$. Now define $\varphi_5 : K - e_3 \to P$ by

$$\varphi_5 \big|_{L_3} = \psi\varphi_3 \big|_{L_3}$$

and

$$\varphi_5 \big|_{K - (e_3 \cup L_3)} = \varphi_1 \big|_{K - (e_3 \cup L_3)}.$$

The function $\varphi_5: K - e_3 \to P$ is injective since $\varphi_1(C_4) \cap D_1 = \emptyset$ and $C_4 \subset K - (e_3 \cup L_3)$ is connected so $\varphi_1(K - (e_3 \cup L_3)) \cap D_1 = \emptyset$. Hence $\varphi_5: K - e_3 \to P$ is an embedding. $\varphi_5(K - e_3) \cap D_2 \subset \varphi_5(L_4 \cup E - e_3)$ and e_2 and e_3 are the only edges in $L_4 \cup E$ which intersect the arc $A \cup e_2$ at a vertex other than the endpoints of $A \cup e_2$, $\{v, v_4\}$. Hence φ_5 can be extended to an embedding of K in P by embedding e_3 in a neighborhood of $\varphi_5(e_2 \cup A)$ in cls D_2.

Case 3. If $\varphi(C_i)$ is contractible for $i = 3$ and 4, then $\varphi_1 |_{C_3 \cup C_4 \cup E - e_3}$ extends to an embedding $\bar{\varphi}_1: C_3 \cup C_4 \cup E \to P$ which contradicts Lemma 5.3 (v). Hence the result follows.

Lemma 5.5. *Let $K \in I(P)$ be a graph which contains disjoint θ graphs but which does not contain a θ graph disjoint from a k-graph. Let v be a vertex of K such that $K -$* st *v is planar. Let L_4 be a subgraph of K which is minimal with respect to $v \in L_4$ and L_4 satisfies Lemma* 5.3 (i) *Then $L_4 \cap$* st *v* contains no more than 6 edges.

Proof. The notation of the statement of Lemma 5.3 will be assumed and used in this proof. Assume $L_4 \cap$ st v contains at least 7 edges. Let $\{e_i \mid 1 \leq i \leq 7\} \subset L_4 \cap$ st v be seven distinct edges indexed such that $e_1 \cup e_4 \subset C_4$ and $\varphi_4(e_i \cup e_{i+1})$ is in the boundary of a component, D_i, of $S^2 - \varphi_4(L_4)$ for $i = 1, 2, 4, 5, 6$. For $i = 1, \ldots, 7$ let v_i be the vertex in e_i with $v_i \neq v$. It remains to show that there is an arc A in $E \cup C_4$ which intersects C_4, contains 2 vertices of C_3, and is disjoint from a θ subgraph, L_5 of L_4 containing v, since this would contradict the minimality of L_4 as the proof of Lemma 5.2 could be reconstructed from the disjoint θ graphs L_1 and L_5 instead of L_1 and L_2. The remainder of the proof will be split in two cases.

Case 1. Assume there is an arc A_1 in $L_4 - (e_3 \cup e_6)$ from v_3 to v_6 disjoint from C_4. Then there is a θ graph L_5 such that $A_1 \cup e_3 \cup e_6 \cup e_7 \subset L_5 \subset L_4$ and $L_5 \cap C_4 = v$. Since K is 3-connected there are two edges in E which do not contain v and which intersect C_3 at distinct vertices, so there exists an arc $A \subset C_4 \cup E$ containing these two edges with $v \notin A$. Hence the result for Case 1.

Case 2. Assume every arc in $L_4 - e_3 \cup e_6$ from v_3 to v_6 intersects C_4. For $i = 3$ or 6, there exist two arcs from v_i to C_4 which do not contain e_i or v and which intersect only at v_i. Since L_4 embeds in S^2 with v_3, v_6 not separated in S^2 by C_4, for $i = 3$ and 6, there exist arcs A_i from v_i to C_4 with $A_3 \cap A_6 = \emptyset$. For $i = 3$ and 6, let A_{i+1} be the arc in C_4 from v to $A_i \cap C_4$ which does not contain e_{i-2}. For $i = 3$ or 6, if two edges of E intersect C_3 at distinct points and intersect A_{i+1} but do not intersect $\{v, A_i \cap C_4\}$, the end points of A_{i+1}, then there is an arc A containing these two edges of E such that $A \cap C_4 \subset A_{i+1}$ and $A \cap L_5 = \emptyset$ for L_5 the θ graph such that $e_{i-2} \cup e_{i-1} \cup e_i \subset L_5$ and $\varphi_4(L_5)$ is the boundary of $D_{i-2} \cup D_{i-1}$, and hence the result. Thus assume for $i = 3$ and 6, there is at most one vertex v_{i+4} in C_3 such that an edge in E contains v_{i+4} and intersects A_{i+1} but not its set of end points $\{v, A_i \cap C_4\}$. Observe that $A_4 \cap A_7$ contains an arc so for each edge e of

$E-\mathrm{st}\,v$, there is $\{i,j\}=\{3,6\}$ with $v_{i+4}\in e$ and $e\cap A_4\cap A_7\subset\{v, A_j\cap C_4\}$. Hence there exists an embedding $\varphi:(E-\mathrm{st}\,v)\cup C_3\cup C_4\to P$ with $\varphi(C_3)$ and $\varphi(C_4)$ contractible and $C_3\cup v$ in the boundary of one of the components of $P-\varphi(E-\mathrm{st}\,v)\cup C_3\cup C_4)$ so φ extends to $E\cap\mathrm{st}\,v$ as an embedding of $E\cup C_3\cup C_4$ into P which contradicts Lemma 5.3 (v). Hence the result follows.

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